

ON EXTENDED EIGENVALUES AND EXTENDED EIGENVECTORS OF TRUNCATED SHIFT

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ABSTRACT. We give a complete description of the set of extended eigenvectors of truncated shifts defined on the model spaces $K_u^2 := H^2 \ominus uH^2$, in the case of u is a Blaschke product.

1. INTRODUCTION AND PRELIMINARIES

Let H be a complex Hilbert space, and denote by $\mathcal{L}(H)$ the algebra of all bounded linear operators on H . If T is an operator in $\mathcal{L}(H)$, then a complex number λ is an extended eigenvalue of T if there is a nonzero operator X such that $TX = \lambda XT$. We denote by the symbol $\sigma_{ext}(T)$ the set of extended eigenvalues of T . The set of all extended eigenvectors corresponding to λ will be denoted as $E_{ext}(\lambda)$. Obviously $1 \in \sigma_{ext}(T)$ for any operator T . Indeed, one can take X being the identity operator.

Let T in $\mathcal{L}(H)$, and let $\sigma(T)$ and $\sigma_p(T)$ denote the spectrum and the point spectrum of T respectively. By a theorem of Rosenblum [4], it was established in [2] that

$$(1.1) \quad \sigma_{ext}(T) \subset \{\lambda \in \mathbb{C} : \sigma(T) \cap \sigma(\lambda T) \neq \emptyset\}.$$

Moreover, when H is finite dimensional, in [2] the set of extended eigenvalues has been characterized by the following theorem

Theorem 1.1. *Let T be an operator on a finite dimensional Hilbert space H . Then $\sigma_{ext}(T) = \{\lambda \in \mathbb{C} : \sigma(T) \cap \sigma(\lambda T) \neq \emptyset\}$.*

Proof. First we consider the case when T is not invertible. In this situation both T and T^* have nontrivial kernels. Let X' be a nonzero operator from kernel of T^* to kernel of T . Define $X = X'P$ where P denotes the orthogonal projection on kernel of T^* . Clearly, $X \neq 0$, and $TX = 0 = \lambda XT$ for any $\lambda \in \mathbb{C}$. Consequently, $\sigma_{ext}(T) = \mathbb{C}$. On the other hand, since T is not invertible, for any complex number λ , $0 \in \sigma(T) \cap \sigma(\lambda T)$. Thus

$$\sigma_{ext}(T) = \mathbb{C} = \{\lambda \in \mathbb{C} : \sigma(T) \cap \sigma(\lambda T) \neq \emptyset\}.$$

Now assume that T is invertible so that $0 \notin \sigma(T)$. In view of (1.1) it suffices to show that $\{\lambda \in \mathbb{C} : \sigma(T) \cap \sigma(\lambda T) \neq \emptyset\} \subset \sigma_{ext}(T)$. So suppose that α is a (necessarily nonzero) complex number such that $\alpha \in \sigma(T)$ and $\alpha \in \sigma(\lambda T)$. Since $\alpha \in \sigma(T)$ there exists a vector a such that $Ta = \alpha a$. On the other hand, $\alpha \in \sigma(\lambda T)$ implies that $\lambda \neq 0$ so $\alpha/\lambda \in \sigma(T)$. Therefore, $\overline{(\alpha/\lambda)} \in \sigma(T^*)$ and there is a vector b such that $T^*b = \overline{(\alpha/\lambda)}b$. Let $X = a \otimes b$. Then $TX = \lambda XT$ and consequently $\lambda \in \sigma(T)$. \square

From this theorem it derives the following consequences

Corollary 1.2. *Let T be an operator on a finite dimensional Hilbert space H . Then*

- (1) *If T is invertible then $\sigma_{ext}(T) = \{\alpha/\beta : \alpha, \beta \in \sigma(T)\}$, and if $Ta = \alpha a$, $T^*b = \overline{\beta}b$ then $a \otimes b \in E_{ext}(\alpha/\beta)$.*
- (2) *$\sigma_{ext}(T) = \{1\}$ if and only if $\sigma(T) = \{\alpha\}, \alpha \neq 0$.*

- (3) $\sigma_{ext}(T) = \mathbb{C}$ if and only if $0 \in \sigma(T)$. Moreover, this assertion remains available in infinite dimensional Hilbert spaces if $0 \in \sigma_p(T) \cap \sigma_p(T^*)$.

The next section contains the needed background on the spaces K_u^2 .

2. BACKGROUND ON K_u^2

Nothing in the section is new, and the bulk of it can be found in standard sources, for example [3], [1], [6] and [5].

2.1. Basic notation, model spaces and kernel functions. Let H^2 be the standard Hardy space, the Hilbert space of holomorphic functions in the open unit disk $\mathbb{D} \subset \mathbb{C}$ having square-summable Taylor coefficients at the origin. We let S denote the unilateral shift operator on H^2 . Its adjoint, the backward shift, is given by

$$(2.1) \quad S^*f(z) = \frac{f(z) - f(0)}{z}.$$

For the remainder of the paper, u will denote a non-constant inner function. the subspace $K_u^2 = H^2 \ominus uH^2$ is a proper nontrivial invariant subspace of S^* , the most general one by the well-known theorem of A. Beurling. The compression of S to K_u^2 will be denoted by S_u . Its adjoint, S_u^* , is the restriction of S^* to K_u^2 . For λ in \mathbb{D} , the kernel function in H^2 for the functional of evaluation at λ will be denoted by k_λ ; it is given explicitly by

$$(2.2) \quad k_\lambda(z) = \frac{1}{1 - \overline{\lambda}z}.$$

Letting P_u denote the orthogonal projection from L^2 onto K_u^2 . The kernel function in K_u^2 for the functional of evaluation at λ will be denoted by k_λ^u . It is natural that k_λ^u equals $P_u k_\lambda$, i.e.,

$$(2.3) \quad k_\lambda^u(z) = \frac{1 - \overline{u(\lambda)}u(z)}{1 - \overline{\lambda}z}.$$

2.2. Riesz bases of K_u^2 . It is known that the model space K_u^2 is finite dimensional if and only if u is finite Blaschke product

$$(2.4) \quad B(z) = \prod_{i=1}^n b_{\alpha_i}^{p_i}, \quad \text{with } b_\lambda = \frac{\lambda - z}{1 - \overline{\lambda}z} \text{ for } \lambda \in \mathbb{D}, \quad p_i, n \in \mathbb{N}^*, \text{ and } \alpha_i \neq \alpha_j \text{ for } i \neq j.$$

In the general case, if B is an infinite Blaschke product defined by

$$(2.5) \quad B(z) = \prod_{i=1}^{\infty} \frac{|\alpha_i|}{\alpha_i} b_{\alpha_i}^{p_i}, \quad p_i \in \mathbb{N}^*,$$

then the following Cauchy kernels

$$(2.6) \quad e_{i,l}(z) = \frac{l!z^l}{(1 - \overline{\alpha_i}z)^{l+1}}, \quad \forall i \geq 1, \quad l = 0, \dots, p_i - 1,$$

span the space K_B^2 . In particular, if $p_i = 1$ for i in \mathbb{N}^* , then $e_{i,0}$ will be denoted by e_i , i.e.,

$$(2.7) \quad e_i(z) = k_{\alpha_i}^B(z).$$

If we denote by $\{e_{i,l}^* : i \geq 1, \quad l = 0, \dots, p_i - 1\}$ (see [5]) the dual set of $\{e_{i,l} : i \geq 1, \quad l = 0, \dots, p_i - 1\}$, (i.e., the set of kernels verifying

$$(2.8) \quad \langle e_{i,k}^*, e_{j,l} \rangle = \delta_{ij} \delta_{kl}, \quad \forall i, j \geq 1, \quad k = 0, \dots, p_i - 1, \quad l = 0, \dots, p_j - 1,$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in L^2 , and δ_{ij} denotes the well-known Kronecker δ -symbol), then we have the following lemma

Lemma 2.1. *If B is a Blaschke product defined by (2.5), then*

$$S_B^* e_{i,l} = \begin{cases} \overline{\alpha_i} e_{i,0} & \text{if } l = 0 \\ l e_{i,l-1} + \overline{\alpha_i} e_{i,l} & \text{otherwise,} \end{cases}$$

and

$$S_B e_{i,l}^* = \begin{cases} \alpha_i e_{i,p_i-1}^* & \text{if } l = p_i - 1 \\ \alpha_i e_{i,l}^* + (l+1) e_{i,l+1}^* & \text{otherwise.} \end{cases}$$

Proof. For the first equality, if $l = 0$, then

$$S_B^* e_{i,0}(z) = \frac{k_{\alpha_i}^B(z) - k_{\alpha_i}^B(0)}{z} = \frac{\overline{\alpha_i}}{1 - \overline{\alpha_i} z} = \overline{\alpha_i} e_{i,0}(z).$$

Otherwise,

$$\begin{aligned} S_B^* e_{i,l}(z) &= \frac{l! z^{l-1}}{(1 - \overline{\alpha_i} z)^{l+1}} = l! \left(\frac{z^{l-1}}{(1 - \overline{\alpha_i} z)^l} + \overline{\alpha_i} \frac{z^l}{(1 - \overline{\alpha_i} z)^{l+1}} \right) \\ &= l e_{i,l-1}(z) + \overline{\alpha_i} e_{i,l}(z). \end{aligned}$$

For the second equality, it is sufficient to use the first one together with the fact that

$$\langle S_B e_{i,k}^*, e_{j,l} \rangle = \langle e_{i,k}^*, S_B^* e_{j,l} \rangle, \quad \forall i, j \geq 1, \quad k = 0, \dots, p_i - 1, \quad l = 0, \dots, p_j - 1.$$

□

If we denote by $E_i = \text{span}\{e_{i,0}, \dots, e_{i,p_i-1}\}$ and by $E_i^* = \text{span}\{e_{i,0}^*, \dots, e_{i,p_i-1}^*\}$, for i in \mathbb{N}^* . Then Lemma 2.1 derives the following consequences

Corollary 2.2. *For each i in \mathbb{N}^* , we have*

- (1) *The subspaces E_i and E_i^* are invariant of S_B^* and S_B respectively.*
- (2) *Let $l \in \{0, 1, \dots, p_i - 1\}$. For each $k = 0, 1, \dots, l$, we have*

$$(S_B - \alpha_i I)^k e_{i,p_i-l-1}^* \neq 0, \text{ and } (S_B - \alpha_i I)^{l+1} e_{i,p_i-l-1}^* = 0.$$

In particular, $\ker(S_B - \alpha_i I)^{l+1} = \text{span}\{e_{i,p_i-l-1}^, \dots, e_{i,p_i-1}^*\}$, and for all $k \geq p_i$, we have $\ker(S_B - \alpha_i I)^k = \ker(S_B - \alpha_i I)^{p_i} = E_i^*$.*

Proof. The first point is trivial. For the second one, we will argue by induction. This result is trivial for $l = 0$. We assume that it is true for all $k = 0, 1, \dots, l-1$, i.e.,

$$x := (S_B - \alpha_i I)^{l-1} e_{i,p_i-l}^* \neq 0, \text{ and } (S_B - \alpha_i I)x = 0.$$

It is enough to show that

$$(S_B - \alpha_i I)^l e_{i,p_i-l-1}^* \neq 0, \text{ and } (S_B - \alpha_i I)^{l+1} e_{i,p_i-l-1}^* = 0.$$

By using Lemma 2.1 and the induction hypothesis, we have that

$$(S_B - \alpha_i I)^l e_{i,p_i-l-1}^* = (p_i - l)x \neq 0,$$

and

$$(S_B - \alpha_i I)^{l+1} e_{i,p_i-l-1}^* = (p_i - 1)(S_B - \alpha_i I)x = 0.$$

Consequently, $\text{span}\{e_{i,p_i-l-1}^*, \dots, e_{i,p_i-1}^*\} \subset \ker(S_B - \alpha_i I)^{l+1}$ and $(S_B - \alpha_i I)^{l+1}$ is injective on $\text{span}\{e_{i,p_i-l-1}^*, \dots, e_{i,p_i-1}^*\}$. To complete the proof, we shall show that $(S_B - \alpha_i I)^{l+1}$ is injective on

$$\text{span}\{E_j^* : j \geq 1 \text{ and } j \neq i\}.$$

But the subspaces E_j^* are invariant of $(S_B - \alpha_i I)^{l+1}$. Thus, it is sufficient to show that $(S_B - \alpha_i I)^{l+1}$ is injective on E_j^* for any $j \neq i$. To do so, suppose to the contrary

that $(S_B - \alpha_i I)^{l+1}x = 0$ for $x \in E_j^*$ and $j \neq i$, then $(S_B - \alpha_i I)^l x \in \text{span}\{e_{i,p_i-1}^*\}$, which contradicts the fact that E_j^* is invariant of $(S_B - \alpha_i I)^l$. \square

Biswas and Petrovic determine in [2] the extended spectrum of truncated shift. Our main result, that is Theorem 3.3, gives a complete description of the set of extended eigenvectors of truncated shift S_B . Moreover, it affirms the result of Biswas and Petrovic for the set $\sigma_{ext}(S_B)$ without using the Sz.-Nagy-Foias commutant lifting theorem. Consequently, it strengthens [2, Theorem 3.10].

3. EXTENDED EIGENVALUES AND EXTENDED EIGENVECTORS OF S_B

If B is a Blaschke product defined by (2.5), it was shown in [3] that $\sigma(S_B) = \overline{\{\alpha_i\}_{i \geq 1}}$, and $\sigma_p(S_B) = \{\alpha_i\}_{i \geq 1}$. For the remainder of this paper, the zeros $\{\alpha_i\}_{i \geq 1}$ are all nonzero. Before showing our main result, we give theorem 3.1 as a direct application of Theorem 1.1 and Lemma 2.1. If B is a finite Blaschke product defined by (2.4) with $p_i = 1$ for all i , then by Corollary 1.2, $\sigma_{ext}(S_B) = \{\alpha_i/\alpha_j : i, j = 1 \dots n\}$ and $e_i^* \otimes e_j \in E_{ext}(\alpha_i/\alpha_j)$. It is natural to ask whether this eigenvector is unique or not. The following theorem answers this question affirmatively.

Theorem 3.1. *If B is a finite Blaschke product defined in (2.4) with $p_i = 1$ for all i , then $\sigma_{ext}(S_B) = \{\alpha_i/\alpha_j : i, j = 1, \dots, n\}$ and $E_{ext}(\alpha_i/\alpha_j) = \text{span}\{e_k^* \otimes e_l : \alpha_k/\alpha_l = \alpha_i/\alpha_j\}$.*

Proof. Since $\{e_i\}_{i=1}^n$ and $\{e_i^*\}_{i=1}^n$ are bases Riesz for K_B^2 , the set $\{E_{ij} := e_i^* \otimes e_j\}_{i,j=1}^n$ is a basis Riesz for $\mathcal{L}(K_B^2)$. Now assume that $X \in \mathcal{L}(K_B^2)$ is a solution to the equation

$$S_B X = \frac{\alpha_i}{\alpha_j} X S_B,$$

then there are a family of complex numbers $\{a_{ij}\}_{i,j=1}^n$ such that

$$S_B \left(\sum_{k,l=1}^n a_{kl} E_{kl} \right) = \frac{\alpha_i}{\alpha_j} \left(\sum_{k,l=1}^n a_{kl} E_{kl} \right) S_B,$$

hence

$$\left(\sum_{k,l=1}^n \frac{\alpha_k}{\alpha_l} a_{kl} E_{kl} \right) S_B = \left(\sum_{k,l=1}^n \frac{\alpha_i}{\alpha_j} a_{kl} E_{kl} \right) S_B,$$

Since S_B is invertible and $\{E_{ij} := e_i^* \otimes e_j\}_{i,j=1}^n$ is a Riesz basis for $\mathcal{L}(K_B^2)$,

$$\frac{\alpha_k}{\alpha_l} a_{kl} = \frac{\alpha_i}{\alpha_j} a_{kl}, \quad \forall k, l = 1, \dots, n,$$

thus

$$E_{ext}\left(\frac{\alpha_i}{\alpha_j}\right) = \text{span}\{e_k^* \otimes e_l : \frac{\alpha_k}{\alpha_l} = \frac{\alpha_i}{\alpha_j}\}.$$

\square

Remark 3.2. *if $\alpha_k/\alpha_l \neq \alpha_i/\alpha_j$ for all $(k, l) \neq (i, j)$, then*

$$E_{ext}\left(\frac{\alpha_i}{\alpha_j}\right) = \{e_i^* \otimes e_j\},$$

that is why we have said that this solution is unique.

Now, let B be an infinite Blaschke product as in (2.5), and let $\{\gamma_i\}_{i \in I}$ be the set of limit points of $\{\alpha_i\}_{i \geq 1}$ on the circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. By (1.1), we have

$$\sigma_{ext}(S_B) \subset \left\{ \frac{\alpha_i}{\alpha_j} : i, j \geq 1 \right\} \cup \left\{ \frac{\alpha_i}{\gamma_j} : i \geq 1, j \in I \right\} \cup \left\{ \frac{\gamma_i}{\alpha_j} : i \in I, j \geq 1 \right\}.$$

The following theorem shows that this inclusion is proper, more precisely

Theorem 3.3. *If B is an infinite Blaschke product defined by (2.5), then*

$$\sigma_{ext}(S_B) = \left\{ \frac{\alpha_i}{\alpha_j} : i, j \geq 1 \right\},$$

and for any $i, j \geq 1$, we have

$$E_{ext}\left(\frac{\alpha_i}{\alpha_j}\right) = \text{span}\left\{ \sum_{k=0}^l \left(\sum_{r=0}^k c_{k-r} \left(\frac{\alpha_m}{\alpha_n} \right)^r \frac{(l+r-k)!(p_m-r-1)!}{(l-k)!(p_m-1)!} e_{m,p_m-r-1}^* \right) \otimes e_{n,l-k} \right\}$$

$\forall m, n \geq 1$ where $\frac{\alpha_m}{\alpha_n} = \frac{\alpha_i}{\alpha_j}$, $l = 0, \dots, \min(p_m - 1, p_n - 1)$, $c_{k-r} \in \mathbb{C}$ and $c_0 \neq 0$.

Proof. Let $\lambda \in \mathbb{C}$ and $X \in \mathcal{L}(K_B^2)$ be such that

$$S_B X = \lambda X S_B,$$

then by Lemma 2.1, for all $j \geq 1$ we have

$$S_B X e_{j,l}^* = \begin{cases} \lambda \alpha_j X e_{j,p_j-1}^* & \text{if } l = p_j - 1 \\ \lambda \alpha_j X e_{j,l}^* + \lambda(l+1) X e_{j,l+1}^* & \text{if } l = 0, \dots, p_j - 2. \end{cases}$$

If $X \neq 0$, then necessarily there are $i, j \geq 1$, l in $\{0, 1, \dots, p_j - 1\}$ and $(c_0 \neq 0)$ in \mathbb{C} such that

$$\lambda = \frac{\alpha_i}{\alpha_j} \text{ and } X e_{j,l}^* = c_0 e_{i,p_i-1}^*.$$

Then

$$(3.1) \quad \begin{aligned} S_B X e_{j,l-1}^* &= \frac{\alpha_i}{\alpha_j} X (\alpha_j e_{j,l-1}^* + l e_{j,l}^*), \\ (S_B - \alpha_i I) X e_{j,l-1}^* &= \frac{\alpha_i}{\alpha_j} l c_0 e_{i,p_i-1}^*, \end{aligned}$$

consequently there exist complex numbers $(c_0^{(1)} \neq 0)$ and c_1 such that

$$X e_{j,l-1}^* = c_0^{(1)} e_{i,p_i-2}^* + c_1 e_{i,p_i-1}^*,$$

moreover, by (3.1)

$$c_0^{(1)} (\alpha_i e_{i,p_i-2}^* + (p_i - 1) e_{i,p_i-1}^*) + c_1 \alpha_i e_{i,p_i-1}^* = \alpha_i (c_0^{(1)} e_{i,p_i-2}^* + c_1 e_{i,p_i-1}^*) + \frac{\alpha_i}{\alpha_j} l c_0 e_{i,p_i-1}^*,$$

hence

$$c_0^{(1)} = \frac{\alpha_i}{\alpha_j} \frac{l}{p_i - 1} c_0.$$

By repeating the same calculation a number of times equal to $\min(p_i - 2, l - 1)$, we obtain that

$$X e_{j,l-k}^* = \sum_{r=0}^k c_{k-r}^{(r)} e_{i,p_i-r-1}^*, \text{ where}$$

$$c_{k-r}^{(r)} = \left(\frac{\alpha_i}{\alpha_j} \right)^r \frac{(l+r-k)!(p_i-r-1)!}{(l-k)!(p_i-1)!} c_{k-r}, \quad k = 2, \dots, \min(p_i - 1, l),$$

thus, if $l \geq p_i$, we have

$$(S_B - \alpha_i I) X e_{j,l-p_i}^* = \frac{\alpha_i}{\alpha_j} (l - p_i + 1) \sum_{r=0}^{p_i-1} c_{p_i-1-r}^{(r)} e_{i,p_i-1-r}^*, \text{ where } c_0^{(p_i-1)} \neq 0,$$

therefore

$$(S_B - \alpha_i I)^{p_i} X e_{j,l-p_i}^* \neq 0 \text{ and } (S_B - \alpha_i I)^{p_i+1} X e_{j,l-p_i}^* = 0,$$

and that contradicts Corollary 2.2. Thus, if $\lambda = \frac{\alpha_i}{\alpha_j}$ and $X \neq 0$, then l must be in the range $\{0, 1, \dots, \min(p_i - 1, p_j - 1)\}$, and the operator

$$X_{i,j} := \sum_{k=0}^l \left(\sum_{r=0}^k c_{k-r} \left(\frac{\alpha_i}{\alpha_j} \right)^r \frac{(l+r-k)!(p_i-r-1)!}{(l-k)!(p_j-1)!} e_{i,p_i-r-1}^* \right) \otimes e_{j,l-k}$$

where $c_{k-r} \in \mathbb{C}$, $c_0 \neq 0$ and $l = 0, \dots, \min(p_i - 1, p_j - 1)$, ,

is a nonzero solution of

$$(3.2) \quad S_B X = \frac{\alpha_i}{\alpha_j} X S_B.$$

Assume that n is a natural number different from j (i.e., $\alpha_n \neq \alpha_j$). Now, we find the image of $e_{n,l}^*$ for $l = 0, 1, \dots, p_n - 1$, under the operator X that verify (3.2), hence

$$S_B X e_{n,l}^* = \begin{cases} \frac{\alpha_i}{\alpha_j} \alpha_n X e_{n,p_n-1}^* & \text{if } l = p_n - 1 \\ \frac{\alpha_i}{\alpha_j} \alpha_n X e_{n,l}^* + \frac{\alpha_i}{\alpha_j} (l+1) X e_{n,l+1}^* & \text{if } l = 0, \dots, p_n - 2. \end{cases}$$

therefore, once again by Corollary 2.2, if there is l in $\{0, 1, \dots, p_n - 1\}$ such that $X e_{n,l}^* \neq 0$, then necessarily there is a natural number m (necessarily different from i) such that

$$\frac{\alpha_i}{\alpha_j} = \frac{\alpha_m}{\alpha_n}, \text{ and } X e_{n,l}^* = c_0 e_{m,p_m-1}^*, \quad (c_0 \neq 0) \in \mathbb{C}.$$

So, in this case, X has the same behavior like the $e_{j,l}^*$ case, i.e., $X = X_{m,n}$ is a solution of (3.2).

Thus, we have exactly described the solution of (3.2) on a set which spans the space K_B^2 . Consequently, $E_{ext}(\alpha_i/\alpha_j)$ is given by

$$E_{ext}\left(\frac{\alpha_i}{\alpha_j}\right) = \text{span}\{X_{m,n}, \forall m, n \geq 1 \text{ where } \frac{\alpha_m}{\alpha_n} = \frac{\alpha_i}{\alpha_j}\},$$

as desired. □

4. CONCLUDING REMARKS

We finish this paper with some remarks which are summarized in the following. First, it is clear that Theorem 3.1 is a particular case of last theorem, nevertheless we have proved it as a direct result of Theorem 1.1.

In addition, if the set of zeros $\{\alpha_i\}_{i \geq 1}$ satisfies the well-known Carleson condition (see [3]), then the set $\{e_{i,l}^*\}$ forms a Riesz basis for K_B^2 , and the solution of 3.2 is given in terms of this basis and the dual Riesz basis $\{e_{i,l}\}$.

Also, if we suppose that $\alpha_0 = 0$ is a zero of B , then by using the proof of Theorem 1.1, we have that $\sigma_{ext}(S_B) = \mathbb{C}$. Indeed, the operator $X = e_{0,p_0-1}^* \otimes e_{0,0}$ satisfies that $S_B X = 0 = \lambda X S_B$, for all λ in \mathbb{C} .

And finally, as a direct result of (2) in Corollary 1.2, if

$$B(z) = b_\alpha^n, \text{ where } n \in \mathbb{N}^* \text{ and } \alpha \in \mathbb{D},$$

then $\sigma_{ext}(S_B) = \{1\}$ and

$$E_{ext}(1) = \text{span}\left\{ \sum_{k=0}^l \left(\sum_{r=0}^k c_{k-r} \frac{(l+r-k)!(n-r-1)!}{(l-k)!(n-1)!} e_{\alpha,n-r-1}^* \right) \otimes e_{\alpha,l-k}, \right. \\ \left. l = 0, \dots, n-1 \right\}.$$

Lastly, this paper gives a complete description of the set of extended eigenvectors of S_u in the case of u is a Blaschke product, and this leads naturally to the following question

Problem 1. *What is the set of extended eigenvectors of S_u in the case of u is a singular inner function?*

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